SOLUTIONS TO
DREW'S
CONIC SECTIONS
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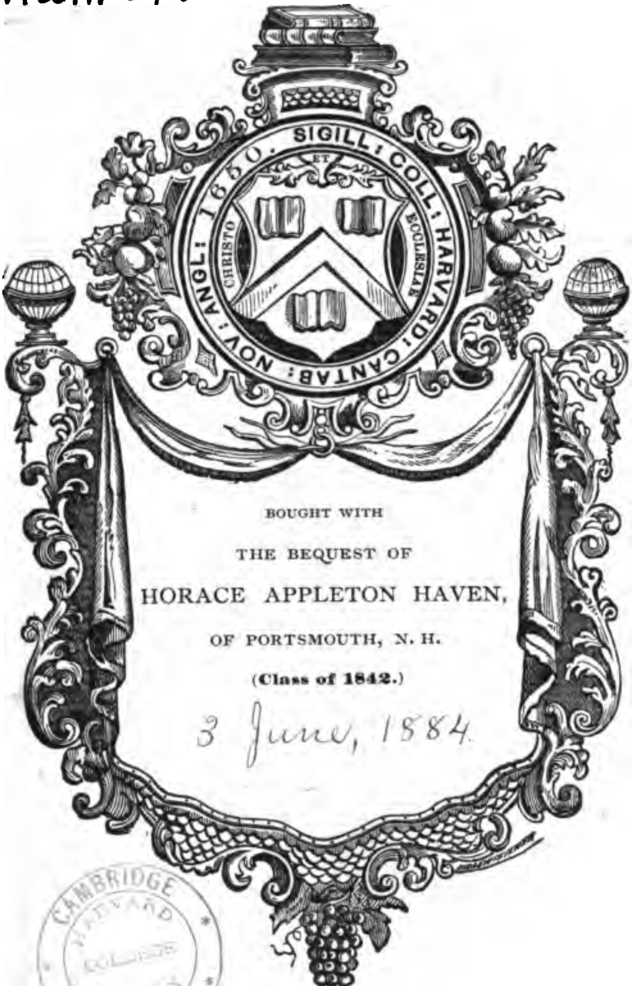
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SOLUTIONS TO PROBLEMS

CONTAINED IN

A GEOMETRICAL TREATISE ON CONIC SECTIONS.

BY THE

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PREFACE.

THE present work is designed to be a continuation of, and is intended to be used by the learner in connexion with, my treatise on "Geometrical Conic Sections." The object is not so much to furnish a key, as to illustrate the application of the principles of Geometry to the solution of questions in Conic Sections. It has been too much the custom to despise the use of geometrical methods, and to reject them as uncertain except in the solution of very elementary problems. The present work will, I hope, show that Geometry is a more effective and practical instrument in the treatment of this subject than is generally supposed.

Another object that I have likewise kept in view is to supply a connecting link by which the student may be led from the simple process of following the steps of a demonstration, to devising for himself the solution of an original geometrical problem. Most persons engaged in teaching must have felt the want of some intervening step to bridge over this difficulty. With this purpose the Solutions are all based on one or other of the propositions in the former treatise, and the student is left to modify the construction and

figure according to the particular question under consideration. An amount of care and attention will thus be rendered necessary, which cannot fail to insure the complete understanding of the principles of the subject, and which will make it impossible for the learner to acquire an apparent but treacherous knowledge, by the undue use of a retentive memory. Prob. 31 of the Parabola, Prob. 55 of the Ellipse, and Prob. 32 of the Hyperbola, afford good illustrations of the plan I have pursued, which really indicates the way in which the properties were themselves suggested.

It has been found convenient to refer occasionally to a *new edition* of the "Conic Sections," which is now in course of publication. In this edition such slight modifications, as experience has shown to be needed, will be introduced. A considerable number of *unsolved* problems will be added, and the work will be brought completely up to the requirements of the present time.

I am indebted for much valuable help in the preparation of these "Solutions," to Mr. A. Freeman, of St. John's College, and to Mr. F. R. Drew, of Sidney College.

W. H. DREW.

BLACKHEATH PROPRIETARY SCHOOL,
November 20th, 1861.

CONIC SECTIONS.

PROBLEMS ON THE PARABOLA.

1. Draw BQ at right angles to AB; then

$$AS \,.\, SQ = BS^2 = 4AS^2,$$
$$\therefore\; SQ = 4AS, \text{ and } AQ = 5AS.$$

2. The triangles PNG, GPK are equiangular, and have the side PG common,

$$\therefore \text{ they are equal in all respects,}$$
$$\therefore\; PK = NG = 2AS.$$

3. $SP = SG = 2NG = 4AS = BC.$

4. Produce PQ to meet the axis in T, and join AQ; then,

$$ST \,.\, SA = SQ^2 = 4AS^2,$$
$$\therefore\; ST = 4AS, \text{ and } SN = 2AS,$$
$$\therefore\; SA : SQ :: SN : SP,$$
$$\therefore\; \angle SQA = \angle SPN, \text{ or } \angle QSB = \angle PSB.$$

5. Since PSZ is a right angle, and SY perpendicular to PZ,

$$\because\; PY \,.\, PZ = SP^2,$$
$$\text{also } PY \,.\, YZ = SY^2 = SA \,.\, SP.$$

6. $$AN \,.\, NL = PN^2 = 4AS \,.\, AN,$$
$$\therefore\; NL = 4AS.$$

7. Let the tangent at P meet the latus rectum in L; then

$$ZY : YL :: XA : AS,$$

$$\therefore\ ZY = YL,$$

and SY is perpendicular to ZL,

$$\therefore\ SZ = SL.$$

8. Since $AT = AN$, and the angles at A are right angles,

$$\therefore\ TY = NY.$$

Also, since the angles at N and Y are right angles, a circle may be described about $SNPY$,

$$\therefore\ TP \,.\, TY = TN \,.\, TS.$$

9. Let the tangent at P meet the tangent at A in Y; then since SYP is a right angle, the circle described about SNP will pass through Y.

Also, since $AY^2 = AS \,.\, AT = AS \,.\, AN$,

$\therefore\ AY$ is a tangent to the circle.

Now $AY : PN :: AT : TN$,

$$\therefore\ AY = \tfrac{1}{2} PN.$$

10. Let the tangent, ordinate, and normal at P' meet the axis in T', N' and G'; then

since $AN' = AT'$ and $N'N = NG'$,

$$\therefore\ AN = \tfrac{1}{2}\, T'G'.$$

Now $PN^2 = 2AS \,.\, 2AN = N'G' \,.\, T'G' = P'G'^2$,

$$\therefore\ PN = P'G'.$$

11. From any point R on the tangent at P draw the tangent RQ; then the triangles SQR, SPR are similar,

$\therefore$ the angle SRQ = the angle SPR,

$\therefore$ the $\angle\ SRQ$ is independent of the point R.

12. Draw NQ a tangent to the circle.

$$\text{Now } RN^2 : A'B^2 :: AN^2 : AA'^2,$$

$$\text{and } PN^2 : A'B^2 :: AN : AA',$$

$$:: AN \,.\, AA' : AA'^2,$$

$$\therefore\ RN^2 - PN^2 : A'B^2 :: AN \,.\, A'N : AA'^2,$$

$$\text{or } RP \,.\, RP' : NQ^2 :: A'B^2 : AA'^2.$$

13. Let B be the point where the axis of the parabola is to touch the circle, and C the other given point.

Draw the tangents TC, TBG intersecting in T; also draw CN perpendicular to TG, and CG passing through the centre O of the circle; then

NT and NG will be respectively the subtangent and subnormal of the parabola required.

Bisect NT in A, and TG in S; then A and S will be respectively the vertex and focus of the parabola.

14. Let QT be the tangent at the point Q.

Through Q draw the chord QOQ'; and let WRO be drawn parallel to the axis meeting QT in W, the parabola in R, and the chord QQ' in O.

Bisect QQ' in V, and draw the diameter TPV parallel to the axis, and join SP.

Now from similar triangles

$$\begin{aligned} QO : OW &:: QV : VT, \\ &:: QV^2 : QV \,.\, VT, \\ &:: 4SP \,.\, PV : 2QV \,.\, PV, \end{aligned}$$

since $QV^2 = 4SP \,.\, PV$ (*Prop.* XV.)

and $VT = 2PV$ (*Prop.* XIV.)

$$\begin{aligned} \therefore\ QO : OW &:: 4SP : 2QV, \\ &:: 4SP : QQ', \end{aligned}$$

$$\therefore\ QO \,.\, QQ' = 4SP \,.\, OW.$$

But $QO \,.\, OQ' = 4SP \,.\, RO$ (*Prop.* XVII.),

$$\therefore\ QQ' : OQ' :: OW : RO,$$

$$\therefore\ QO : OQ' :: WR : RO.$$

15. Join SP; then

$$PV = ST = SP = PO.$$

16. Draw VM at right angles to the axis; then

$$\text{since } VM : MS :: PN : NT,$$
$$\therefore\ MS = NT.$$
$$\text{But } PN^2 = 2AS \,.\, NT,$$
$$\therefore\ VM^2 = 2AS \,.\, SM,$$

$\therefore$ the locus of V is a parabola, whose vertex is S, and latus rectum half that of the given parabola.

17. Produce QV, UR to meet the tangent at P in W and X; then as in *Prop.* XIX. it can be proved that WV and XU are each equal to $4\,SP$,

$\therefore$ WV and XU are equal and parallel,

$\therefore$ UV is parallel to the tangent at P.

18. By similar triangles

$$CR : CA :: AN : PN,$$
$$\therefore\ PN \,.\, CR = AC \,.\, AN,$$
$$\text{or } PN^2 = AC \,.\, AN,$$

$\therefore$ the locus of P is a parabola whose axis is AB, and latus rectum equal to the given distance AC.

19. Let CP and AR intersect in V; and draw VN $\perp^r$ to AC, and VM $\perp^r$ to the tangent at A; then

$$\begin{aligned} VN : AN &:: CR : CA, \\ &:: CR : CP, \\ &:: VN : CV, \end{aligned}$$
$$\therefore\ CV = AN = VM,$$

$\therefore$ the locus of V is a parabola whose focus is C, and directrix AM.

20. Let OQ, OQ' be two equal tangents drawn from the point O in the axis; and let them be cut by the tangent RPR' in R and R'.

Join SP; SQ, SQ'; SR, SR'; then

$$\text{the } \angle\, SOR' = \text{the } \angle\, SOQ = \text{the } \angle\, SQO;$$

$$\begin{aligned}\text{and the } \angle\, SR'O &= \text{the supplement of } SR'Q', \\ &= \ldots\ldots\ldots\ SPR' \ (\textit{Prop.}\ \text{XII.}) \\ &= \text{the angle } SPR, \\ &= \text{the angle } SRQ \ (\textit{Prop.}\ \text{XII.})\end{aligned}$$

$\therefore$ in the triangles SOR', SQR,

the angles SOR', $SR'O = SQR$, SRQ each to each,

$$\text{and } SO = SQ,$$

$$\therefore\ RQ = OR',$$

$$\text{and } OQ = OQ',$$

$$\therefore\ OR = R'Q'.$$

21. Join QE, $Q'E$; then

$$\text{the } \angle\, QEQ' = 2\, \angle\, QOQ'.$$

Again, let OQ and OQ' meet the axis ASx in T and T'; then

$$\begin{aligned}\angle\, QSx &= SQT + STQ, \\ &= 2\, \angle\, STQ = 2\, \angle\, QOV;\end{aligned}$$

$$\text{so the } \angle\, Q'Sx = 2\, \angle\, ST'Q' = 2\, \angle\, Q'OV,$$

$$\therefore \text{the } \angle\, QSQ' = 2\, \angle\, QOQ',$$

$$\therefore \text{the } \angle\, QSQ' = \text{the } \angle\, QEQ',$$

$\therefore$ the circle described about the triangle QEQ' will pass through S.

If QQ' pass on the left of S, a *similar* proof will hold good.

It must then be shown that the angles QSQ', QEQ' are together equal to two right angles.

22. Draw the ordinates PN, pn; then

$$\begin{aligned}SN : Sn &:: SP : Sp, \\ &:: XN : Xn,\end{aligned}$$

$$\therefore\ SN : XN :: Sn : Xn,$$
$$\therefore\ XN + SN : XN - SN :: Xn + Sn : Xn - Sn,$$
$$\therefore\ 2AN : 2AS :: 2AS : 2An,$$
$$\text{or } AN : AS :: AS : An,$$
$$\therefore\ 4AS.AN : 4AS^2 :: 4AS^2 : 4AS.An;$$
$$\text{or } PN^2 : 4AS^2 :: 4AS^2 : pn^2,$$
$$\therefore\ PN : 2AS :: 2AS : pn,$$
$$\text{or } PN : 4AS :: AS : pn.$$

Again,

$$\begin{aligned} SQ : AS &:: PN : AN, \\ &:: 4AS.PN : 4AS.AN, \\ &:: 4AS.PN : PN^2, \\ &:: 4AS : PN, \\ &:: pn : AS, \end{aligned}$$
$$\therefore\ SQ = pn,$$
$$\text{so } Sq = PN.$$

23. Join QS and produce it to q; then

$$\left.\begin{aligned} &SR \text{ bisects the angle } PSq, \\ \text{and } &Sr \ \ldots\ldots\ldots\ldots\ pSq, \end{aligned}\right\} (\textit{Prop.}\ \text{IV.})$$

$\therefore$ the angle RSr is half of the two PSq and pSq,

$\therefore$ the angle RSr is a right angle,

$$\therefore\ DR.Dr = SD^2 = 4AS^2.$$

24. Join PQ, and draw OV parallel to the axis; then

since $OQ' : OQ :: PV : QV$,

$\therefore$ $Q'P$ is parallel to OV,

$\therefore$ the $\angle\, OQP$ = the $\angle\, QOV$,

= the $\angle\, QTS$, if OQ meet the axis in T,

= the $\angle\, SQO$,

and the $\angle\, OPQ'$ = the $\angle\, OPS$,

= the $\angle\, SOQ$ (*Prop.* XII.)

$$\therefore \text{ the triangle } POQ' \text{ is similar to } OSQ,$$
$$\therefore PQ' : OP :: OQ : OS,$$
$$\therefore OS . PQ' = OQ . OP.$$

25. Draw QV parallel to the tangent at P; and draw SY perpendicular to the tangent, and join AY; then

$$\begin{aligned} QD : QV &:: SY : ST, \\ &:: SY : SP, \\ \therefore QD^2 : QV^2 &:: SY^2 : SP^2, \\ &:: SP . SA : SP^2, \\ &:: SA : SP, \\ &:: 4AS : 4SP, \\ &:: 4AS . PV : 4SP . PV, \end{aligned}$$
$$\therefore QD^2 = 4AS . PV.$$

26. Draw the tangent PT at the point P meeting the axis in T; then by Prop. 14

$$TA : AO :: PO : OQ.$$
$$\text{But } AT = AM,$$
$$\therefore AM : AO :: PO : OQ;$$
$$\text{So } AN : AO :: OQ : OP,$$
$$\therefore AM : AO :: AO : AN,$$
$$\therefore AM . AN = AO^2.$$

27. By similar triangles

$$AM : PM :: QN : AN,$$
$$\therefore AM^2 : PM^2 :: QN^2 : AN^2,$$
$$\text{or } AM^2 : 4AS . AM :: 4AS . AN : AN^2,$$
$$\text{or } AM : 4AS :: 4AS : AN,$$
$$\therefore AM . AN = (4AS)^2.$$

28. Let Q be the centre of the circle inscribed in the sector OAP.

Draw OQR, meeting the circles at their point of contact R.

Complete the quadrant AOB, and draw BN parallel to OA; and through Q draw MQN, meeting OA and BN at right angles in M and N; then

$$QR = QM,$$
$$\text{and } OR = MN,$$
$$\therefore OQ = QN,$$

$\therefore$ the locus of Q is a parabola, whose focus is O, and directrix BN.

29. (1.) $AN : AM :: PN^2 : QM^2,$

$$:: AN^2 : SM^2;$$
$$\text{or } AN^2 : AN . AM :: AN^2 : SM^2,$$
$$\therefore SM^2 = AN . AM.$$

(2.) Bisect QQ' in V, and draw VR parallel to the axis bisecting AP in R; then

$$AS + AM = SQ$$
$$AS + AM' = SQ',$$
$$\therefore MM' = SQ - SQ',$$
$$= 2\,SV = 2\,AR,$$
$$= AP.$$

30.

$$RV : Q'V' :: PV : PV',$$
$$:: QV^2 : Q'V'^2,$$
$$RV . Q'V' : Q'V'^2 :: QV^2 : Q'V'^2,$$
$$\therefore RV . Q'V' = QV^2;$$
$$\text{So } R'V' . QV = Q'V'^2,$$
$$\therefore RV : R'V' :: QV^3 : Q'V'^3.$$

31. In the fig. of Prop. XVII. let $Q'q'$ be the diameter of the circle, and Qq the chord joining the two other points of intersection of the circle and parabola.

Join VV' meeting the axis in E; VV' is at right angles to Qq.

Now since the rectangle $Q'O \,.\, Oq = Q'O \,.\, Oq'$,

$\therefore$ PV and $P'V'$ are equally distant from the axis,

$$\therefore\ VE = V'E.$$

Let Qq, $Q'q'$ meet the axis in F and F'; and draw VM, $V'M'$ at right angles to the axis; then

$$FF' = MM' = 2EM = 2 \text{ subnormal of } P = 4AS.$$

32. Draw OPV parallel to the axis bisecting QQ' in V; then since the tangent at P is parallel to QQ', it will be at right angles to OQ, and will therefore meet OQ on the directrix. Let this point be Z; then

$$OZ : ZQ :: OP : PV,$$
$$\therefore\ OZ = ZQ.$$

33. The parabolas will intersect at the extremities B, C, of the latus rectum. Draw the normal BG; then the angle at which the parabolas intersect is twice the angle SBG, which is a right angle, since $SG = 2AS = SB$.

34. Let BSB' be the chord of the circle of curvature at B through the focus. Bisect BB' in C, and draw CO at right angles to BB', meeting the diameter of curvature in O; then since CO is parallel to the axis,

$$BO : BG :: BC : BS,$$
$$\text{and } BC = 2BS, \quad \therefore\ BO = 2BG.$$

35. Draw PW parallel to the axis; then

$$\text{the angle } SPF = WPH = SHP,$$

and the angle at S is common to the triangles SPF, SHP;

$\therefore$ these triangles are similar,

$$\therefore\ SF : SP :: SP : SH,$$
$$\therefore\ SF \,.\, SH = SP^2 = SG^2.$$

36. Let QPQ' be the inscribed triangle.

Draw the tangents QO, $Q'O$, meeting in O; and the tangent tPt' meeting OQ, OQ' in t and t'.

Produce $Q'P$, QP to meet OQ, OQ' in q and q'.

Join $q'q$; then what we have to prove is that qq', and the tangent at P will meet $Q'Q$ produced in the same point.

Draw OV, PM, qm, $q'm'$, tn, $t'n'$, parallel to the axis of the parabola, meeting QQ' in V, M, m, m', n, n'. Produce MP to meet OQ in R, and let qm, $q'm'$ meet tPt' in r and r'.

The problem will clearly be solved if we can show that

$$qr : qm :: q'r' : q'm'.$$

Now
$$qm : Qm :: RM : QM,$$
$$Q'm : qm :: Q'M : PM,$$
$$\therefore\ Q'm : Qm :: RM \,.\, Q'M : QM \,.\, PM;$$
$$\text{but } RM : PM :: QQ' : Q'M \text{ (Prob. 14.)}$$
$$\therefore\ Q'm : Qm :: QQ' \,.\, Q'M : QM \,.\, Q'M,$$
$$:: QQ' : QM,$$
$$\therefore\ Qm : QQ' :: QM : QQ' + QM.$$

Again,
$$qr : PR :: rt : Pt,$$
$$:: 2mn : 2Mn,$$
$$:: 2Qm - 2Qn : 2Mn,$$
$$:: 2Qm - QM : QM \text{ (since } tn \text{ bisects } QM);$$
$$\text{but } 2Qm : QM :: 2QQ' : QQ' + QM,$$
$$\therefore\ 2Qm - QM : QM :: QQ' - QM : QQ' + QM,$$
$$\therefore\ qr : PR :: QQ' - QM : QQ' + QM,$$
$$\text{and } PR : PM :: QM : Q'M,$$
$$\therefore\ qr : PM :: QM : QQ' + QM,$$
$$:: Qm : QQ',$$
$$\therefore\ qr : Qm :: PM : QQ'.$$

In the same manner it can be shown that

$$q'r' : Q'm' :: PM : QQ',$$
$$\therefore\ qr : q'r' :: Qm : Q'm'.$$

But
$$\begin{aligned} Qm : qm &:: QV : OV, \\ &:: Q'm' : q'm', \\ \therefore\ Qm : Q'm' &:: qm : q'm', \\ \therefore\ qr : q'r' &:: qm : q'm', \\ \text{or } qr : qm &:: q'r' : q'm'. \end{aligned}$$

37. Let the tangent RR' meet the curve in P; and draw Rm, PM, $R'm'$ parallel to the axis meeting QQ' in m, M and m'.

Now
$$\begin{aligned} OR : RQ &:: Vm : Qm, \\ &:: 2Vm : 2Qm, \\ &:: 2QV - 2Qm : 2Qm, \\ &:: QQ' - QM : QM, \\ &:: Q'M : QM. \end{aligned}$$

Similarly
$$\begin{aligned} OR' : R'Q' &:: QM : Q'M, \\ \therefore\ OR : RQ &:: R'Q' : OR'. \end{aligned}$$

38. Let AQ and AQ' be the chords, and R the further angle of the rectangle.

Join QQ' and draw AR bisecting it in V. Also draw VP parallel to the axis, and at P draw the tangent PT parallel to QQ'.

Draw PN, VM, RL at right angles to the axis.

Make $AO' = 2AO = 8AS$ (*Probs.* 26 *and* 27);

Then $RL = 2VM = 2PN$,

and
$$\begin{aligned} O'L &= AL - AO' = 2AM - 2AO, \\ &= 2OM = 2TN = 4AN. \end{aligned}$$

But $PN^2 = 4AS \,.\, AN$,

$\therefore\ 4PN^2 = 4AS \,.\, 4AN$.

or $RL^2 = 4AS \,.\, O'L$,

$\therefore$ the locus of R is a parabola equal to the original one with its vertex at O'.

39. Draw AH at right angles to the tangent at P; then if PK be the chord of curvature through A,

$$PK : PU :: AH : AP \text{ (See fig. } \textit{Prop.} \text{ XIX.)}$$
$$\therefore PK . AP = PU . AH.$$

Again, $PU : 4SP :: SP : SY$ (*Prop.* XIX.)

$$:: TY : AY,$$
$$\therefore PU : PY :: 4SP : AY \text{ (since } PY = TY).$$

Also, $AH : PY :: AY : SP,$

$$\therefore PU . AH : PY^2 :: 4SP . AY : SP . AY,$$
$$\therefore PU . AH = 4PY^2.$$
$$\therefore PK . AP = 4PY^2.$$

40. Let QQ' be a chord. Bisect it in V, and draw QM, VK, $Q'M'$ at right angles to the directrix; then

$$2VK = QM + Q'M'$$
$$= SQ + SQ'.$$

But $SQ + SQ' > QQ'$, unless QQ' pass through S,

$$\therefore VK > QV.$$

Hence the circle described upon QQ' as diameter will only reach the directrix when the chord QQ' passes through the focus, in which case it will touch the directrix, since the angle VKM is a right angle.

41. Let A and A' be the vertices of the parabolas.

Produce the axes to meet in T, and draw the common tangent TP.

Join SS', cutting PT at right angles in Y.

Join AY, and draw $S'X$ perpendicular to ST.

Now since AY is perpendicular to SX,

$$\therefore\ SY : S'Y :: SA : AX.$$

But
$$SY = S'Y,$$
$$\therefore\ SA = AX,$$
$$\therefore\ S' \text{ is on the directrix.}$$

42. In the fig. of *Prop.* XIX. let Q' be the other point where the circle meets the parabola, and draw $Q'X'R'$ parallel to the axis, meeting the circle in X' and the tangent at P in R'; then

XX' is parallel to RPR' (*Prob.* 17.)

and when the circle becomes the circle of curvature,

WX' is parallel to RPR',

$\therefore$ WX' is at right angles to PU.

Now if the points Q' and U coincide,

$X'U$ and PW will be equal,

and WX' will bisect PU,

and $PWUX'$ will be a square,

$$\therefore \text{ the } \angle\, GPS = \text{the } \angle\, GPW,$$
$$= \text{the } \angle\, GPN,$$

$\therefore$ P is at the extremity of the latus rectum.

PROBLEMS ON THE ELLIPSE.

1. DRAW the diameter CD parallel to the tangent at P, meeting $S'P$ in E. Also draw the normal PGF perpendicular to CD, and meeting it in F.

Then the angle $SPS' = 2$ the angle $S'PG$.

Now since PE is always equal to AC, it is evident that the angle $S'PG$ will be greatest when PF is least.

But $PF \,.\, CD = AC \,.\, BC$ (*Prop.* XXII. *Cor.*)

$\therefore$ PF is least when CD is greatest, that is, when $CD = AC$ and P is at the extremity B of the minor axis.

2. Draw the latus rectum LSL'; then

$$AC^2 : BC^2 :: AS \,.\, A'S : SL^2 \text{ (Prop. XIII.)}$$
$$\text{or } AC^2 : BC^2 :: AC^2 - CS^2 : SL^2 \text{ (Prop. IV.)}$$
$$\text{or } AC^2 : BC^2 :: BC^2 : SL^2,$$
$$\text{or } AC : BC :: BC : SL,$$
$$\therefore AA' : BB' :: BB' : LL',$$
$$\therefore LL' \text{ is a 3d proportional to } AA' \text{ and } BB'.$$

3. Draw BH parallel to LS, meeting the major axis in H; then

$$\triangle BCH : \triangle LSS' :: BC^2 : SL^2$$
$$:: AC^2 : BC^2. \text{ (Prob. 2.)}$$

Take $CK = \frac{1}{2} CH$; then

$$\text{rectangle } BB' \,.\, CK = \triangle BCH,$$
$$\therefore BB' \,.\, CK : \triangle LSS' :: AA'^2 : BB'^2.$$

4. Let the tangent at L to one of the ellipses meet the minor axis produced in t; then

$$Ct \,.\, SL = BC^2 \text{ (\textit{Prop.} XIV.)}$$
$$\therefore\ Ct : BC :: BC : SL$$
$$:: AC : BC \text{ (\textit{Prob.} 2.)}$$
$$\therefore\ Ct = AC.$$

Hence Ct is constant, and the tangents will all pass through t.

5. In this problem the ellipses are to have their major axes equal as well as a common focus.

Let S be the given focus, and YPY' the given fixed line (see fig. *Prop.* XV.); then since S and YPY' are both given in position, the point Y of the perpendicular SY will be a fixed point.

Now $YC = AC$ (*Prop.* XV.)

Hence the locus of C is a circle whose centre is Y and radius AC.

6. Draw SY, $S'Y'$ perpendicular to the given line (see fig. *Prop.* XV.), and take BC a mean proportional between SY and $S'Y'$; then

BC is the semi-minor axis. (*Prop.* XV.)

Also, since $AC^2 = BC^2 + CS^2$,

AC is also known; and since

$CX : CA :: CA : CS$ (*Prop.* II.)

the position of the directrix is known, and the ellipse may be described.

7. See fig. *Prop.* IX.

$$TQ : PN :: AT : AN,$$
$$TQ' : PN :: A'T : A'N,$$

Again,
$$CT \,.\, CN = CA^2,$$
$$\therefore\ CT : CA :: CA : CN,$$

$$\therefore\ CT + CA : CT - CA :: CA + CN : CA - CN,$$
$$\text{or } A'T : AT :: A'N : AN,$$
$$\therefore\ A'T : A'N :: AT : AN,$$

Hence $$TQ : PN :: TQ' : PN,$$
$$\therefore\ TQ = TQ'.$$

8. See fig. *Prop.* IV.

If SBS' be a right angle, SC and CB are equal,
$$\therefore\ AC^2 = SB^2 = SC^2 + BC^2 = 2BC^2.$$

9. Draw the ordinates QM, PN; then
$$QM^2 : AM^2 :: PN^2 : CN^2,$$
$$\therefore\ AM \,.\, A'M : AM^2 :: AC^2 - CN^2 : CN^2,$$
$$\text{or } A'M : AM :: AC^2 - CN^2 : CN^2,$$
$$\therefore\ 2AC : AM :: AC^2 : CN^2,$$
$$\therefore\ 2AO : AQ :: AO^2 : CP^2,$$
$$\text{or } 2AO^2 : AO \,.\, AQ :: AO^2 : CP^2,$$
$$\therefore\ AO \,.\, AQ = 2CP^2.$$

10. If AB and CD are equally inclined to the axis, the diameters parallel to them will also be so, and will be therefore equal. Hence if AB and CD or these lines produced meet in O,
$$OA \,.\, OB = OC \,.\, OD.$$

Hence a circle may be described about A, B, C, D; and if AC and BD or these lines produced intersect in O',
$$O'A \,.\, O'C = O'B \,.\, O'D,$$
$\therefore$ AC and BD are equally inclined to the axis.

So with regard to AD and BC.

11. Draw PM, pm at right angles to the directrix. Produce MP, QS to meet in Q', and mp, qS to meet in q'; then

$$Q'P : PM :: SA : AX,$$

$$\therefore\ Q'P = SP,$$

$$\therefore\ SQ' \text{ bisects the angle } PSA';$$

$$\text{So } Sq' \qquad\qquad\qquad pSA',$$

$$\therefore\ Q'Sq' \text{ is a right angle.}$$

$$\therefore \text{ also } QSq \text{ is a right angle.}$$

12. Draw the diameter SK cutting the tangent at B at right angles in O; then

$$SK \,.\, SO = SB^2,$$

$$\text{or } SK \,.\, BC = AC^2.$$

13. Through Q draw RQR' touching the inner circle in Q, and cutting the outer in R and R'; then

$$\begin{aligned} PQ \,.\, P'Q &= QR^2, \\ &= CR^2 - CQ^2, \\ &= AC^2 - BC^2 = CS^2. \end{aligned}$$

14. Draw the ordinate PN, and the semi latus rectum SL'; then by similar triangles

$$\begin{aligned} SL : SN &:: SG : SP, \\ &:: SA : AX; \end{aligned}$$

Also $$SP : XN :: SA : AX,$$

$$\therefore\ PL : SX :: SA : AX;$$

But $$SL' : SX :: SA : AX,$$

$$\therefore\ PL = SL'.$$

15. Draw the normals PG, DG'; then

$$PG^2 : CD^2 :: BC^2 : AC^2 \ (\textit{Prop.}\ \text{XXIV.})$$

and $$DG'^2 : CP^2 :: BC^2 : AC^2,$$

$$\therefore\ PG^2 + DG'^2 : CP^2 + CD^2 :: BC^2 : AC^2.$$

But $$CP^2 + CD^2 = AC^2 + BC^2.$$

$$\therefore\ PG^2 + DG'^2 : AC^2 + BC^2 :: BC^2 : AC^2,$$

$$\therefore\ PG^2 + DG'^2 \text{ is constant.}$$

16. Produce PG to meet CD in F, and join CP; then

$$CQ^2 = CP^2 + PQ^2 - 2PF \cdot PQ \text{ (\textit{Euclid}, II. 13)}$$
$$= CP^2 + CD^2 - 2PF \cdot CD,$$
$$= AC^2 + BC^2 - 2AC \cdot BC \text{ (\textit{Props.} XX. XXII.)}$$
$$\therefore CQ = AC - BC.$$

17. Let the tangents at P and P' intersect at right angles in O. Draw CK, CK' at right angles respectively to OP and OP'; then

$$CO^2 = CK^2 + CK'^2.$$

Again, from S draw SY and SY' perpendicular respectively to OP and OP'; and join CY, CY'. Also let SY' and CK intersect in H; then

$$CK^2 + SH^2 = CY^2 = AC^2,$$
$$\text{and } CK'^2 + CH^2 = CY'^2 = AC^2,$$
$$\therefore CK^2 + CK'^2 + CS^2 = 2AC^2,$$
$$\therefore CK^2 + CK'^2 = AC^2 + BC^2,$$
$$\therefore CO^2 = AC^2 + BC^2.$$

18. Draw the ordinate QM; then

$$NR : AN :: QM : AM,$$
$$\text{and } NS : A'N :: QM : A'M,$$
$$\therefore NR \cdot NS : AN \cdot A'N :: QM^2 : AM \cdot A'M.$$

But $$PN^2 : AN \cdot A'N :: QM^2 : AM \cdot A'M.$$
$$\therefore PN^2 = NR \cdot NS.$$

19. By similar triangles

$$GL : PN :: SG : SP,$$
$$:: CS : CA \text{ (\textit{Prop.} XI.)}$$

20. Since the tangent at L meets the directrix in X,

$$QN : XN :: SL : SX,$$
$$:: SA : AX;$$

Also $$SP : XN :: SA : AX,$$
$$\therefore QN = SP.$$

21. See fig. *Prop.* VII.

$$PZ \text{ bisects the } \angle SPW,$$
$$\therefore \text{ the } \angle SPZ \text{ is } > \text{ the } \angle MPZ.$$

22. See fig. *Prop.* XXIV.

Let $S'P$ meet CD in E. Produce PF to meet the **axis** minor in K, and join EK; then

$$PK : PG :: NC : NG,$$
$$:: AC^2 : BC^2 \text{ (Prop. XII.)}$$
$$\therefore PK \,.\, PF : PF \,.\, PG :: AC^2 : BC^2,$$
$$\therefore PK \,.\, PF = AC^2 \text{ (Prop. XXIV.)}$$
$$= PE^2 \text{ (Prop. XV. Cor.)}$$
$$\therefore \text{ the } \angle PEK \text{ is a right angle.}$$

So also if PS produced meet CD' in E', and $E'K$ be joined, the angle $PE'K$ may be proved to be a right angle.

$\therefore$ the perpendiculars EK and $E'K$ meet the minor axis in the same point.

23.
$$NG : NC :: BC^2 : AC^2 \text{ (Prop. XII.)}$$
$$\therefore CT \,.\, NG : CT \,.\, CN :: BC^2 : AC^2,$$
$$\therefore CT \,.\, NG = BC^2 \text{ (Prop. IX.)}$$
$$\therefore CT : BC :: BC : NG.$$

24. A circle may be described about $PYSN$;

$$\therefore \angle PNY = \angle PSY,$$
So
$$\angle PNY' = \angle PS'Y',$$
$$\text{and } \angle PSY = \angle PS'Y',$$
$$\therefore \angle PNY = \angle PNY',$$
$$\therefore PY : PY' :: NY : NY'.$$

25. See fig. *Prop.* XV.

Draw CZ at right angles to the tangent meeting $S'P$ in K; then by similar triangles

$$KE : CE :: SP : PY.$$

But $$CE = PY,$$

$$\therefore KE = SP.$$

26. Let R be the point where the circle described about QPQ' meets the ellipse; join PR meeting QQ' in O, and draw the diameter CP' parallel to PR, and CD parallel to QQ'; then from the ellipse

$$QO \,.\, OQ' : PO \,.\, OR :: CD^2 : CP'^2 \text{ (\textit{Prop.} XXIX.)}$$

and from the circle

$$QO \,.\, OQ' = PO \,.\, OR,$$

$$\therefore CP' = CD,$$

$\therefore$, CP' being fixed in position, PR, which is drawn parallel to CP', meets the ellipse in a fixed point.

27. Produce QP', $Q'P'$ to meet the tangent at P in R and R'. Draw the diameter CD parallel to the tangent at P; also draw the diameters CE, CE' parallel respectively to QP' and $Q'P'$; then it is evident from *Prop.* XXIX. that

$$RP' \,.\, RQ : RP^2 :: CE^2 : CD^2,$$

$$\text{and } R'P' \,.\, R'Q' : R'P^2 :: CE'^2 : CD^2.$$

But $$CE^2 = CE'^2,$$

Since $P'Q$ and $P'Q'$ are equally inclined to the axis,

$$\therefore RP' \,.\, RQ : R'P' \,.\, R'Q' :: RP^2 : R'P^2.$$

But $$RP : R'P :: RP' : R'P',$$

since $P'P$ bisects the $\angle\, Q'P'R$,

$$\therefore RP' \,.\, RQ : R'P' \,.\, R'Q' :: RP'^2 : R'P'^2,$$

$$\therefore RQ : R'Q' :: RP' : R'P',$$

$\therefore$ QQ' is parallel to RR'.

28. See fig. *Prop.* XXII.

Let a parallelogram $LPlL'l'$ be formed by drawing tangents at the extremities of a pair of conjugate diameters PCP', and DCD'.

On the portion of the ellipse DPD', take any point Q, and at the extremities of the diameter QCQ' draw the tangents RQr, $R'Q'r'$ meeting the tangents at D and D' in R, R' and r, r' respectively. Also let Rr intersect Ll in O.

Now, since Ll is bisected in P, it is evident, that on whichever side of P the point O is taken, the parallelogram Rr' is greater than the parallelogram LL', and that the excess is twice the difference of the unequal triangles ROl, rOL.

Hence the parallelogram formed by drawing tangents at the extremities of conjugate diameters (which is, of constant area) is the least which can be described about an ellipse.

29. $(CP + CD)^2 = CP^2 + CD^2 + 2CP \cdot CD,$

$$= AC^2 + BC^2 + 2CP \cdot CD \text{ (Prop. XX.)}$$

Hence $(CP + CD)^2$ is least when $CP \cdot CD$ is least.

Now $CP \cdot CD$ is $> PF \cdot CD$, unless $PF = CP$,

and $PF \cdot CD = AC \cdot BC$ (*Prop.* XXII. *Cor.*)

Hence $CP \cdot CD$ is $> AC \cdot BC$, unless $PF = CP$;

and when $PF = CP$,

$$CP \cdot CD = AC \cdot BC;$$

$\therefore$ $CP \cdot CD$ is least, when PF and CP are equal, or when CP and CD coincide with the axes of the ellipse.

30. Let S be the given focus, and P the given point. Join SP, and on SP produced make SW equal to the major axis AA'.

With centre P and radius PW, describe a circle WU. The other focus of the ellipse must be on this circle.

Now, since in an ellipse

$$CS^2 = AC^2 - BC^2 \text{ (Prop. IV.)}$$

$$\therefore SS'^2 = AA'^2 - BB'^2.$$

Hence we have the following construction for finding S' the other focus and centre.

On SW as diameter describe a semicircle, and in it place WH equal to the minor axis BB'.

With centre S and radius SH describe a circle cutting the circle WU in S'.

Then $$SS'^2 = SH^2 = SW^2 - WH^2$$
$$= AA'^2 - BB'^2.$$
$$\therefore\ S' \text{ is the other focus.}$$
Bisect SS' in C; then C is the centre.

31. Let Q be any point on the ellipse, and let QP produced meet CD in M, and let QP' meet CD in N. Also draw QV parallel to CD, and QR parallel to CP.

Then $$CM : CP :: QV : PV,$$
$$CN : CP' :: QV : P'V,$$
$$\therefore\ CM \,.\, CN : CP^2 :: QV^2 : PV \,.\, P'V,$$
$$:: CD^2 : CP^2 \ (\textit{Prop.}\ \text{XXI.})$$
$$\therefore\ CM \,.\, CN = CD^2.$$

32. Let CP and CD be conjugate diameters.

Then $$SP : XN :: SA : AX \ (\text{See fig. } \textit{Prop.}\ \text{IX.})$$
$$:: CA : CX \ (\textit{Prop.}\ \text{II.})$$
$$\therefore\ SP : AC :: XN : CX,$$
$$\therefore\ AC - SP : AC :: CN : CX,$$
$$\text{or } AC - SP : CN :: AC : CX.$$
So also if DN' be the ordinate of D,
$$AC - SD : CN' :: AC : CX,$$
$$\therefore\ (AC - SP)^2 + (AC - SD)^2 : CN^2 + CN'^2 :: CA^2 : CX^2,$$
$$:: CS^2 : CA^2;$$
But $$CN^2 + CN'^2 = CA^2 \ (\textit{Prop.}\ \text{XX.})$$
$$\therefore\ (AC - SP)^2 + (AC - SD)^2 = CS^2.$$

33. Draw the ordinates PN, DR, and produce them to meet the auxiliary circle in P', D'; also let CB produced meet the circle in B'.

Join $B'D'$, $B'P'$; $A'P'$, AD', and let $A'P'$, AD' intersect in O'.

Now, since $P'D'$ is a quadrant of a circle,

$$\text{the arc } AP' = \text{the arc } B'D',$$
$$\therefore B'P' \text{ is parallel to } AD';$$

So also $\quad B'D'$ is parallel to $A'P'$,

$$\therefore B'O' \text{ is a parallelogram.}$$

And since BP and $B'P'$ produced meet the axis in the same point, and $B'P'$ is parallel to AD',

$$\text{and } BC : B'C :: DR : D'R,$$
$$\therefore BP \text{ is parallel to } AD;$$

So also $\quad BD$ is parallel to $A'P$,

$$\therefore OB \text{ is a parallelogram.}$$

Again, join PD, $P'D'$. These lines will, when produced, meet the axis in the same point. Now if any straight line be drawn at right angles to AA', meeting PD, $P'D'$, it is easily seen that the portions of this line intercepted by the triangles PBD, $P'B'D'$ respectively are in the proportion of BC to AC.

Hence the triangles PBD, $P'B'D'$ are also in this proportion.

$$\therefore ODBP : O'D'B'P' :: BC : AC.$$

Now since $P'D'$ is a quadrant, the triangle $P'B'D'$ will have its greatest value when B' is half way between P' and D', for the perpendicular from B' on $P'D'$ will then be greatest.

Hence $ODBP$ is greatest when CP and CD are equal.

34. $PS \,.\, Sp : AS \,.\, A'S :: CQ^2 : AC^2$ (*Prop.* XXIX.)

$$\begin{aligned} \therefore PS \,.\, Sp : CQ \,.\, Cq &:: AS \,.\, A'S : AC^2, \\ &:: CA^2 - CS^2 : AC^2, \\ &:: BC^2 : AC^2. \end{aligned}$$

35. See fig. *Prop.* XIX.

$$AT : AC :: PN : CN,$$
$$:: CR : DR \quad (Prop.\ \text{XIX}.\ Cor.)$$
$$:: AC : At,$$
$$\therefore AT \,.\, At = AC^2.$$

36. Draw the diameter Cp, Cq, Cr parallel to the tangents; then

$$P'R^2 : P'Q^2 :: Cr^2 : Cq^2 \quad (Prop.\ \text{XXIX.})$$
$$\text{or } P'R : P'Q :: Cr : Cq.$$

So $$Q'P : Q'R :: Cp : Cr,$$
$$\text{and } R'Q : R'P :: Cq : Cp,$$
$$\therefore P'R \,.\, Q'P \,.\, R'Q = P'Q \,.\, Q'R \,.\, R'P.$$

37. See fig. *Prop.* IX.

Let the tangent at P meet the tangent at A in Q; then

QP and QA subtend equal angles at S (*Prop.* XVII.)

$\therefore$ QS bisects the $\angle\, TSP$,

and PT bisects the $\angle\, SPW$,

$\therefore$ Q is the centre of the circle ST, SP and $S'P$ produced,

$\therefore$ the locus required is the tangent at the vertex A.

38. It is easily seen that this question is only the converse of Prob. 33.

39. Let ACA', aCa' be the major axes of the ellipses, inclined to one another at any angle.

Let P be the point where the ellipses intersect within the angle ACa, and Q the point within the angle $A'Ca$; then

since CP is a common semi-diameter of both ellipses,

$\therefore$ CP makes equal angles with CA and Ca,

$\therefore$ CP bisects the angle ACa,

So CQ bisects the angle $A'Ca$,

$\therefore$ the $\angle\, PCQ$ is half of ACa and $A'Ca$,

$\therefore$ PCQ is a right angle.

40. Draw the ordinates PM, QN; then

$$PM : QN :: SP : SQ,$$
$$:: XM : XN,$$
$$\therefore PM : XM :: QN : XN,$$
$$\therefore \text{the } \angle PXM = \text{the } \angle QXN.$$

41. Draw the ordinates PM, QN; and let Cp, Cq be the diameters parallel to the tangents at P and Q respectively; then

$$RP \cdot R'P : RQ \cdot R'Q :: XM \cdot X'M : XN \cdot X'N.$$

Now $XM : SP :: CA : CS :: XN : SQ$,

and $X'M : S'P :: CA : CS :: X'N : S'Q$,

$$\therefore XM \cdot X'M : SP \cdot S'P :: XN \cdot X'N : SQ \cdot S'Q,$$
$$\therefore RP \cdot R'P : RQ \cdot R'Q :: SP \cdot S'P : SQ \cdot S'Q,$$
$$:: Cp^2 : Cq^2 \text{ (\textit{Prop.} XXIV.)}$$
$$:: OP^2 : OQ^2 \text{ (\textit{Prop.} XXIX.)}$$

42. See fig. *Prop.* XIV.

Let the normal to the ellipse at P meet CQ produced in R'; then

the triangles $R'PR$ and TPQ are similar.

$$\therefore R'R : TQ :: PR : QP,$$
$$:: CQ : TQ,$$
$$\therefore R'R = CQ = AC,$$
$$\therefore CR' = CR + AC,$$
$$= BC + AC \text{ (\textit{Prop.} XIV.)}$$

$\therefore$ the locus of R' is a circle, whose centre is C and radius $AC + BC$.

43. See fig. *Prop.* XV.

Let CD meet SP produced in E, and $S'P$ in E'; then

$$S'E' = S'P - PE' = S'P - AC,$$
$$\text{and } SE = PE - SP = AC - SP.$$

But $$S'P + SP = 2AC,$$
$$\therefore\ S'P - AC = AC - SP,$$
$$\therefore\ SE = S'E'.$$

Again, if D and D' be the diameters of the circles described about SCE and $S'CE'$, since the perpendiculars from S and S' upon CD are equal,

$$\therefore\ D : D' :: CS \,.\, SE : CS' \,.\, S'E' \text{ (*Euclid* VI. C.)}$$
$$\therefore\ D = D'.$$

44. Let PSP' be any focal chord, and O its middle point. Join CO and produce it to meet the ellipse in Q. Also draw the tangent QT, and the ordinate QN.

Since O is the middle point of PP',
$$\therefore\ QT \text{ is parallel to } PP',$$
$$\therefore\ OM : MC :: QN : NC,$$
$$\text{and } OM : MS :: QN : NT,$$
$$\therefore\ OM^2 : MC \,.\, MS :: QN^2 : NC \,.\, NT,$$
$$:: BC^2 : AC^2 \text{ (Prop. XIII.)}$$

$\therefore$ the locus of O is an ellipse, similar to the given ellipse, described upon CS as major axis.

45. Let the circle cut the minor axis in K; then since the angle BSK is a right angle,

$$\therefore\ BK \,.\, BC = SB^2,$$
$$= AC^2.$$

But if BU be the chord of the circle of curvature at B through C, and therefore in this case the diameter,

$$BU \,.\, BC = 2AC^2 \text{ (Prop. XXVI.)}$$
$$\therefore\ 2BK = BU,$$
$$\therefore\ BK = BO,$$

$\therefore$ K coincides with the centre of the circle of curvature.

46. See fig. *Prop.* XI.

Draw SO bisecting the angle PSS', and meeting PG in O; then O is the centre of the circle.

Draw the ordinate OM at right angles to AC.

$$\begin{aligned}
\text{Now}\quad MG : NG &:: OG : PG, \\
&:: SG : SG + SP, \\
&:: CS : CS + AC \ (\textit{Prop.}\ \text{XI.}) \\
\text{Also}\quad NG ; NC &:: CA^2 - CS^2 : AC^2 \ (\textit{Prop.}\ \text{XII.}) \\
\therefore MG : NC &:: (CA - CS)\ CS : AC^2, \\
&:: CA \,.\, CS - CS^2 : AC^2, \\
\text{and } Cq : NC &:: CS^2 : AC^2, \\
\therefore MC : NC &:: CA - CS : AC^2, \\
&:: CS : AC, \\
\therefore MC : CS &:: NC : AC, \\
\therefore S \quad : CS &:: AN : AC, \\
\text{and } S'M : CS &:: A'N : AC, \\
\therefore SM - S'M : CS^2 &:: AN - A'N : AC^2, \\
&:: PN^2 : BC^2 \ (\textit{Prop.}\ \text{XIII.}) \\
\therefore SM \,.\, S'N : PN^2 &:: CS^2 : CA^2 - CS^2. \\
\text{Also}\quad PN^2 : OM^2 &:: (CA + CS)^2 : CS^2, \\
\therefore SM \,.\, S'M : OM^2 &:: (CA + CS)^2 : (CA - CS)^2, \\
&:: A'S^2 : AS^2.
\end{aligned}$$

Hence the locus of O is an ellipse, whose major is SS'.

Also if Cb be the semi-minor axis,

$$Cb : CS :: AS : A'S.$$

47. Let P, P', Q, Q' be the four points of intersection.

Join PQ, $P'Q'$, and let these lines or these produced intersect in O.

Draw the semi-diameters Cp, Cp' parallel respectively to PQ and $P'Q'$.

Then $PO \,.\, OQ : P'O \,.\, OQ' :: Cp^2 : Cp'$ (*Prop.* XXIX.)

But $PO \cdot OQ = P'O \cdot OQ'$ from the circle,

$$\therefore\ Cp = Cp'.$$

48. Let OP, OP' be the two lines at right angles to each other, and C the centre of the ellipse in any position; then, as in Prob. 17,

$$OC^2 = AC^2 + BC^2,$$

$\therefore$ the locus of C is the arc of a circle whose centre is O.

49. See fig. *Prop.* XXIV.

Draw SM at right angles to CD, and let MS and CP be produced to meet in Q. Also draw QH at right angles to the axis.

Now
$$\begin{aligned} CH : CN &:: CQ : CP, \\ &:: CM : CF, \\ &:: CS : CG, \\ \therefore\ CH : CS &:: CN : CG, \\ &:: CX : CS\ (\textit{Prop.}\ \text{XII.}) \\ \therefore\ CH &= CX, \end{aligned}$$

$\therefore$ the point Q is on the directrix.

50. For *radius* read *diameter*.

If O be the centre of the circle of curvature at 1,

$$PO \cdot PF = CD^2\ (\textit{Prop.}\ \text{XXVII.})$$

$$\begin{aligned} \therefore\ PO : CD &:: CD : PF, \\ &:: CD^2 : PF \cdot CD, \\ &:: CD^2 : AC \cdot BC\ (\textit{Prop.}\ \text{XXII. } \textit{Cor.}) \end{aligned}$$

Hence if
$$CD^2 = AC \cdot BC,$$

$$\therefore\ PO = CD,$$

$$\therefore \text{ also } PO^2 = AC \cdot BC,$$

$\therefore$ PO is a mean proportional between AC and BC at the same time that CD is.

Hence, if a circle be described with centre C and radius

a mean proportional between AC and BC, intersecting the ellipse in D, and CP be drawn parallel to the tangent at D, P will be the point required.

51. See *Prop.* XXVI.

If $CP = CD$, then

$$PH \cdot CP = 2\,CP^2,$$

$$\therefore PH = 2\,CP = PP',$$

$\therefore$ the circle of curvature meets the ellipse at the point P.

52. Let the lines drawn at right angles to AP, $A'P$ meet in Q, and draw QM at right angles to AA'; then

$$QM : AM :: AN : PN,$$

$$\text{and } QM : A'M :: A'N : PN,$$

$$\therefore QM^2 : AM \cdot A'M :: AN \cdot A'N : PN^2,$$

$$:: AC^2 : BC^2 \ (\textit{Prop.}\ \text{XIII}.$$

$\therefore$ the locus of Q is an ellipse, similar to and concentric with the given ellipse, and having AA' for its minor axis.

53. Let C and c be the centres of the two ellipses, which intersect one another in the points P, Q, R, S. Join PQ, RS; and let these lines, or these lines produced, intersect one another in the point O.

From C and c draw the semi-diameters CU, CV parallel to PQ, and cu, cv parallel to RS; then

$$PO \cdot OQ : RO \cdot OS :: CU^2 : CV^2 \ (\textit{Prop.}\ \text{XXIX.})$$

$$\text{and } RO \cdot OQ : RO \cdot OS :: cu^2 : cv^2 \qquad (\textit{ditto.})$$

$$\therefore CU : CV :: cu : cv.$$

Now if the ellipses have their major axes at right angles to one another, it is evident that since the angles UCV and ucv are equal, and CU, cu parallel,

$$CU >=< CV,$$

according as $\quad cv >=< cu.$

and $\therefore$ the only case in which the proportion

$$CU : CV :: cu : cv$$

can hold good is when CU and CV are equal.

Again, if the major axes of the ellipses are parallel, the only cases in which the proportion can hold good is when the ellipses are similar, or as before when CU and CV are equal.

But if the ellipses are similar, and their major axes parallel, they can only intersect in two points.

Hence, whether the major axes are at right angles or parallel, we must have

$$CU = CV.$$

$$\therefore PO \, . \, OQ = RO \, . \, OS.$$

And a circle may therefore be described through P, Q, R, S.

54. Let VCV' be the fixed diameter of the circle, along which CN is measured; and draw the diameter DCD' parallel to NQ; then, since NQ makes a constant angle with NP, the diameter DCD' is also fixed.

Now $\quad PN^2 = VN \, . \, V'N$ from the circle,

$$\therefore QN^2 = VN \, . \, V'N,$$

$$\text{or } QN^2 : VN \, . \, V'N :: CD^2 : CV^2.$$

$\therefore$ the locus of Q is an ellipse, of which CV and CD are the *equal* conjugate semi-diameters.

The position and length of the *equal* conjugate diameters being now known, the ellipse may be constructed as follows,

Draw ACA' bisecting the acute angle formed by VCV', DCD', and BCB' bisecting the supplementary angle formed by the same lines.

Then ACA', BCB' represent the *directions* of the major and minor axes of the ellipse respectively.

Draw VM at right angles to CA, and Cv bisecting the angle ACB, and let these lines meet in v; then

the point v is on the *auxiliary* circle.

Take CA equal to Cv, and

make $BC : AC :: VM : vM$;

then AC and BC are the semi-major and minor axes.

55. It is evident from the symmetry that the two ellipses will have a common centre, and that that centre must coincide with the point where the diameters of the parallelogram intersect.

Also since the ellipses are supposed to have their major axes of the same length, their auxiliary circles will also coincide.

Now since any side of the parallelogram is a *common* tangent to both ellipses, and by *Prop.* XV. the feet of the perpendiculars from the foci upon the tangent lie on the circumference of the auxiliary circle, it is evident that the perpendiculars from the foci of one of the ellipses upon the sides of the parallelogram, will coincide with the perpendiculars from the foci of the other ellipse upon the same lines.

Or, in other words, the lines joining the foci of the ellipses will be at right angles to the sides of the given parallelogram. These lines will therefore form a parallelogram equiangular to the given parallelogram.

56. Let H be the common focus of the ellipses, and S, S' their other foci.

Then, since the major axes of the ellipses are equal, any point where they intersect must be equidistant from S and S'.

Any point, therefore, where the ellipses intersect must lie upon the line bisecting SS' at right angles, for this line contains *all* the points, whose distances from S and S' are equal to each other.

And since this line can only intersect either ellipse in *two* points, it is evident that the ellipses themselves can only intersect in two points.

57. Since arc SR = arc $S'R'$,

$\therefore$ PR bisects the angle SPS',

and coincides with the normal PQ produced.

Hence the triangles SPR, QPS' are similar,

$$\therefore SR : PR :: S'Q : S'P,$$
$$:: CS : CA \text{ (\textit{Prop.} XI.)}$$
$$:: CS \,.\, CA : AC^2;$$

But

$$PR : PQ :: MC : NQ,$$
$$:: AC^2 : BC^2 \text{ (\textit{Prop.} XII.)}$$
$$\therefore SR : PQ :: CS \,.\, CA : BC^2.$$

58. Draw SO bisecting the angle PSS', and meeting the normal PQ in O; then O is the centre of the inscribed circle. Draw OM perpendicular to the axis; then

$$OM : PN :: OQ : PQ,$$
$$:: SQ : SQ + SP,$$
$$:: CS : CS + CA,$$
$$\therefore OM : CS :: PM : CS + CA,$$

Also

$$2R : S'P :: SP : PN \text{ (\textit{Euclid} VI. B.)}$$
$$\therefore 2R \,.\, r : CS \,.\, S'P :: SP : CS + CA,$$
$$\text{or } 2R \,.\, r : SP \,.\, S'P :: CS : CS + CA.$$

59. Join GK, GL and let KL meet PG in O; then, since the angles at K and L are right angles, and the angle SPS' is bisected by PG,

$\therefore$ the triangles PGL, PGK are equal in all respects.

Again, since OP, PL are equal to OP, PK,

and the $\angle OPL =$ the $\angle OPK$,

$\therefore OL = OK$,

and the $\angle POL =$ the $\angle POK$,

$\therefore PG$ is at right angles to KL.

60. See fig. *Prop.* XV.

Draw $S'O$ parallel to SP meeting YS produced in O; then

$$S'O : S'P :: S'W : WP,$$
$$:: SP + S'P : SP,$$

$$\therefore\ S'O = SP + S'P = 2AC$$

$\therefore$ the locus of O is a circle, whose centre is S' and radius $2AC$.

61. Draw the conjugate diameter CD parallel to TPt. Also draw the ordinate DR; then

$$TP : CD :: PN : DR,$$
$$:: CR : CN \text{ (Prop. XIX. Cor.)}$$
$$:: CD : Pt,$$
$$\therefore\ TP\,.\,Pt = CD^2.$$

But $$CP\,.\,PL = TP\,.\,Pt,$$
$$\therefore\ 2CP\,.\,PL = 2CD^2,$$
$$= PH\,.\,CP \text{ (Prop. XXVI.)}$$
$$\therefore\ 2PL = PH.$$

Again, $$CP\,.\,CL = CP\,.\,PL + CP^2,$$
$$= CD^2 + CP^2,$$
$$= AC^2 + BC^2 \text{ (Prop. XX.)}$$

62. Since CR' bisects PQ, the tangent at R' will be parallel to PQ (*Prop.* XVIII.)

So the tangents at P' and Q' will be parallel respectively to QR and PR.

$\therefore$ the triangle rqp formed by the tangents at P', Q', R' is similar to the triangle PQR.

Now, since $P'Q'$ is equal and parallel to PQ,

$\therefore$ the triangle $rP'Q'$ = the triangle PQR.

So the triangle $qP'R'$ = the triangle PQR,

and the triangle $pQ'R'$ = the triangle PQR.

$\therefore$ the triangle $pqr = 4$. the triangle PQR.

63. See fig. *Prop.* XV.

Draw the ordinate PN; then PN is a common chord of both circles.

Now $\angle NS'J = \angle NY'J = \angle Y'NP = \angle Y'S'P$,

and $\angle ASI = \angle IYN = \angle YNP = \angle YSP$;

But $\angle YSP = \angle Y'S'P$,

$\therefore \angle NS'J = \angle ASI$,

and $CS' = CS$;

$\therefore$ IS and JS' will, when produced, meet CB' in the same point.

64. Draw the diameter RCR', and from any point Q on the ellipse draw QR and QR'.

Also draw the diameters PCP', DCD' parallel respectively to QR and QR'.

Now since C is the middle point of RR',

$\therefore$ PCP' bisects QR',

$\therefore$ the tangent at P is parallel to QR',

and therefore to DCD';

$\therefore$ CD is conjugate to CP.

65. See fig. *Prop.* XVI.

If SP and SP' are in the same straight line, then the point O is on the directrix (*Prop.* VIII.)

and the $\angle OSP$ is a right angle (*Prop.* VI.)

$\therefore$ the $\angle OMP$ is a right angle (*Prop.* XVII.)

so also the $\angle OM'P'$ is a right angle,

$\therefore$ the angles MOM' and $MS'M'$ are equal to two right angles.

But $MOM' = 2POP'$,

$\therefore 2T + O = 2$ right angles.

66. See fig. *Prop.* XV.

$$\triangle SPS' = \triangle WSS' - WSP,$$
$$= SY \,.\, YY' - SY \,.\, YP.$$
$$= SY \,.\, PY'.$$

Now $$PY' : S'Y' :: PY : SY,$$

$$\therefore SY . PY' : SY . S'Y' :: PY : SY,$$

$$\text{or } \triangle SPS' : BC^2 :: PY : SY.$$

So also if SY_1 be perpendicular to the tangent at Q,

$$\triangle SQS' : BC^2 :: QY_1 : SY_1.$$

But since the angles SPY, SQY, are complementary,

$$\therefore PY : SY :: SY_1 : QY_1;$$

$$\therefore \triangle SPS' : BC^2 :: BC^2 : \triangle SQS'.$$

The angles SPS', SQS' are evidently supplementary.

The least value of either of the angles is zero, and the greatest $\angle SBS'$.

Hence the problem is evidently impossible,

unless $2\,SBS'$ is $\not>$ 2 right angles,

or SBC is $\not> \frac{1}{2}$ a right angle,

in which case $CS \not> BC$,

or $BC \not< CS$.

67. Let CP be the fixed diameter; and CD the diameter which is conjugate to it in any one of the ellipses.

On CP produced take a *fixed* point O, and from it draw the tangent OQ to the ellipse to which CD belongs.

Draw QV parallel to CD, meeting CP in V, and produce PC to meet the ellipse in P'.

Now $$CV . CO = CP^2 \text{ (}Prop.\text{ XVIII.)}$$

$$\therefore CV \text{ is constant,}$$

$$\therefore PV \text{ and } P'V \text{ are also constant.}$$

Again, $$QV^2 : PV . P'V :: CD^2 : CP^2 \text{ (}Prop.\text{ XXI.)}$$

But $$CD^2 = CP^2,$$

$$\therefore QV^2 = PV . P'V,$$

$$\therefore QV \text{ is constant.}$$

And since the point V is also fixed, therefore the locus of Q is a circle whose centre is V, and radius a mean proportional between PV and $P'V$.

68. See fig. *Prop.* IV.

Let AA' be one of the longer sides of the rectangle, and B' the intersection of the diagonals.

Produce AB', $A'B'$ to meet the tangents to the ellipse, at A' and A in R' and R.

Then RR' will be the side of the rectangle opposite to AA'.

Take any point P on the upper portion of the ellipse, and join PR, PR' meeting AA' in M and M'.

Also produce the ordinate NP to meet the auxiliary circle in Q.

By similar triangles

$$MM' : RR' :: PM : PR,$$
$$:: MN : AN,$$
$$\therefore MM' : MN :: AA' : AN.$$

Also
$$MN : AM :: PN : AR,$$
$$:: PN : BB',$$
$$\therefore MM' : AM :: PN \,.\, AA' : AN \,.\, BB',$$
$$:: PN \,.\, AC : AN \,.\, BC.$$

But
$$PN : QN :: BC : AC \text{ (\textit{Prop.} XIII. \textit{Cor.})}$$
$$\therefore PN \,.\, AC = QN \,.\, BC,$$
$$\therefore MM' : AM :: QN \,.\, BC : AN \,.\, BC,$$
$$:: QN : AN.$$

So also
$$MM' : A'M' :: QN : A'N,$$
$$\text{and } AN : QN :: QN : A'N,$$
$$\therefore AM : MM' :: MM' : A'M'.$$

69. See fig. *Prop.* XXIV.

Draw SH perpendicular to PG; then

$$PQ : PF :: SP : SH.$$

And if PM be the ordinate of P,

$$PM : PG :: SH : SG,$$

$$\therefore PQ \,.\, PM : PF \,.\, PG :: SP : SG.$$

$$:: CA : CS. \text{ (\textit{Prop.} XI.)}$$

But $PF \,.\, PG = BC^2$ (*Prop.* XXIII.)

$$\therefore PQ \,.\, PM : BC^2 :: CA : CS,$$

$\therefore PQ$ varies inversely as PM.

70. See fig. *Prop.* XIX.

Draw QM at right angles to AA'; then since the triangles QMS, CNP are similar,

$$\therefore QM : SM :: CN : NP.$$

So also $QM : S'M :: CR : DR,$

$$\therefore QM^2 : SM \,.\, S'M :: CN \,.\, CR : NP \,.\, DR.$$

But $CN : DR :: AC : BC$ (*Prop.* XIX. *Cor.*)

and $CR : NP :: AC : BC,$

$$\therefore CN \,.\, CR : DR \,.\, NP :: AC^2 : BC^2,$$

$$\therefore QM^2 : SM \,.\, S'M :: AC^2 : BC^2.$$

Hence the locus of Q is an ellipse, ***similar*** to and concentric with the given ellipse, and which has SS' for its minor axis.

PROBLEMS ON THE HYPERBOLA.

1. Let S and S' be the centres of the given circles, and P the centre of a circle described touching them both.

Join SP, $S'P$, and let these lines, or these lines produced, meet the circles of which S and S' are the centres in the oints of contact Q and Q'; then

(1) If the given circles are touched *both* internally, or *both* externally,

$$\text{since } PQ = PQ'$$
$$\therefore SP \backsim S'P = SQ \backsim SQ',$$

and the locus of P is one branch of an hyperbola, of which S and S' are the foci; but

(2) If the given circles are touched, *one* externally, and the *other* internally,

$$SP + S'P = SQ + S'Q'$$

and the locus is an ellipse.

2. See fig. *Prop.* VI.

Let the tangent at P meet the tangents at A and A', in U and U'; join SU, SU'; then

since UA and UP are tangents,

$\therefore$ the $\angle\ USA$ = the $\angle\ USP$. (*Prop.* XIV.)

So the $\angle\ U'SA'$ = the supplement of $U'SP$ (*Prop.* XIV.)

= the $\angle\ U'SQ$,

$\therefore$ the $\angle\ USA$ is half the $\angle\ ASP$,

and the $\angle\ U'SA$ is half the $\angle\ ASQ$,

$\therefore$ USU' is a right angle.

$\therefore$ S lies on the circle described upon UU' as diameter. So for the point S'.

3. See fig. *Prop.* XVII.

Join SO, $S'O$; then

$$\text{since } BC^2 = CS^2 - CA^2,$$
$$\therefore AO^2 = AS \,.\, AS',$$
$$\therefore \text{the } \angle SOS' \text{ is a right angle,}$$

which proves the proposition.

4. See fig. *Prop.* XVI.

Let PII' meet the conjugate axis in K; then

$$KI^2 : CK^2 :: AC^2 : AO^2,$$
$$\text{or } KI^2 : PN^2 :: AC^2 : BC^2.$$

But $CN^2 - AC^2 : PN^2 :: AC^2 : BC^2$ (*Prop.* X.)

$$\therefore CN^2 - AC^2 = KI^2,$$
$$\text{or } PK^2 - KI^2 = AC^2.$$

But $$PK^2 - KI^2 = PI \,.\, PI',$$
$$\therefore PI \,.\, PI' = AC^2.$$

5. See fig. *Prop.* VI.

Let the tangent at A meet PT in U, and join US; then

US bisects the $\angle ASP$ (*Prop.* XIV.)

and PT bisects the $\angle SPS'$ (*Prop.* VI.)

$\therefore$ U is the centre of the circle inscribed in the triangle SPS', which proves the proposition.

6. $$PN^2 : MN^2 :: QN^2 : CQ^2$$
$$:: AN \,.\, A'N : AC^2,$$
$$\therefore PN^2 : AN \,.\, A'N :: MN^2 : AC^2.$$

But $PN^2 : AN \,.\, A'N :: BC^2 : AC^2$ (*Prop.* X.)

$$\therefore MN = BC.$$

7. Since $CT \,.\, CN = CA^2$ (*Prop.* VIII.)

$$\therefore CT : CA :: CA : CN,$$
$$:: CP : AQ,$$
$$\therefore AQ \text{ is parallel to } PT.$$

8. If P be one of the points where the ellipse and hyperbola intersect, and S and S' the common foci; then

The line PT which bisects the $\angle\ SPS'$, is a normal to the ellipse, and a tangent to the hyperbola at the point P.

$\therefore$ the curves intersect at right angles.

9.
$$
\begin{aligned}
AR : PN &:: AT : TN,\\
\text{and } A'r : PN &:: A'T : TN,\\
\therefore AR \,.\, A'r : PN^2 &:: AT \,.\, A'T : TN^2,\\
&:: CA^2 - CT^2 : TN^2,\\
&:: CT \,.\, CN - CT^2 : TN^2,\\
&:: CT \,.\, TN : TN^2,\\
&:: CT : TN,\\
&:: CT \,.\, CN : CN \,.\, TN,\\
&:: AC^2 : AN \,.\, A'N \ (\textit{Prop. } \text{X.})\\
\text{or } AR \,.\, A'r : AC^2 &:: PN^2 : AN \,.\, A'N,\\
&:: BC^2 : AC^2 \ (\textit{Prop. } \text{X.})\\
\therefore AR \,.\, A'r = BC^2 &= AR' \,.\, A'r'.
\end{aligned}
$$

10. See fig. *Prop.* XXI.

Let LPl, $L'P'l'$ be the two tangents; then

$$CL \,.\, Cl = CS^2 = CL' \,.\, Cl' \ (\textit{Prop. } \text{XXII.})$$
$$\therefore CL : CL' :: Cl' : Cl,$$
$$\therefore Ll' \text{ is parallel to } L'l.$$

11. See fig. *Prop.* XVII.

Since the $\angle\ CES$ is a right angle,

$$
\begin{aligned}
\therefore SE : CE &:: AO : AC,\\
&:: BC : AC,
\end{aligned}
$$

But $\quad CE = AC$ (*Prop.* XVII.)

$$\therefore SE = BC.$$

12. See fig. *Prop.* XVII.

$$CE : CO :: CX : CA,$$
$$:: CA : CS \text{ (\textit{Prop.} II.)}$$
$$\therefore AE \text{ is parallel to } SO.$$

13. Since $CP^2 \backsim CD^2 = AC^2 \backsim BC^2$ (*Prop.* XXIV.)

$$\therefore \text{ if } AC^2 = BC^2$$
$$CP^2 = CD^2.$$

14. $$NG : NC :: BC^2 : AC^2 \text{ (\textit{Prop.} IX.)}$$
$$\therefore \text{ if } BC^2 = AC^2,$$
$$NG = NC,$$
$$\therefore PG = CP.$$

15. See fig. *Prop.* XXV.

Join QP, QP'; bisect QP, QP' in U and U', and join CU, CU'; then

since QP' is parallel to CU, and QP to CU',

$\therefore$ CU bisects the chords parallel to CU',

and CU' bisects the chords parallel to CU;

$\therefore$ CU and CU' are in the directions of conjugate diameters.

But in a *rectangular* hyperbola the conjugate diameters are equal (Prob. 13), and are therefore equally inclined to either asymptote.

Hence QP and QP', which are parallel to conjugate diameters, are also equally inclined to either asymptote.

16. See fig. *Prop.* XXIII.

Since $CPLD$ is a parallelogram, and that the diameters of a parallelogram bisect each other,

$$\therefore PD \text{ is bisected by } CL.$$

17. See fig. *Prop.* XVI.

Join AP, and produce it both ways to meet the lines drawn through A' parallel to CR and Cr in Q and Q'. Also let QQ' meet CR and Cr in U and U'; then

$$PU = AU', \text{ and } PU' = AU \text{ (\textit{Prop.} XIX.)}$$

But since AA' is bisected in C,

$$QU = AU, \text{ and } Q'U' = AU',$$
$$\therefore PQ = UU', \text{ and } PQ' = UU',$$
$$\therefore PQ = PQ'.$$

18. See fig. *Prop.* XXXII.

$$SP = PI \ (Prop.\ \text{XXXII.})$$
$$= CR - AC \ (Prop.\ \text{XVII.})$$

So also $S'P' = CR - BC$.

$$\therefore S'P' - SP = AC - BC.$$

19. Draw CE parallel to SP, and DE perpendicular to CE; also draw SY and $S'Y'$ perpendicular to the tangent at P; then

the triangles DEC, SPY are similar.

$$\therefore DE : CD :: SY : SP,$$
$$:: S'Y' : S'P \ (Prop.\ \text{XII.})$$
$$\therefore DE^2 : CD^2 :: SY . S'Y' : SP . S'P,$$
$$:: BC^2 : CD^2 \ (Prop.\ \text{XXXII.})$$
$$\therefore DE = BC.$$

20. Let CP, CD be the given conjugate diameters.

Join PD, and bisect PD in H; and join CH.

Also draw CK parallel to PD; then

CH and CK are the asymptotes.

Bisect the angle HCK by the line CA, and draw CB at right angles to CA; then

CA and CB are in the directions of the axes.

Draw the ordinate PN, and the tangent PT parallel to CD, meeting CN in T.

Take CA a mean proportional between CN and CT; then

CA is the transverse semi-axis.

Also produce NP and CH to meet in R, and take BC a fourth proportional to RN, PN, and AC, so that

$$RN : PN :: AC : BC.$$

Then BC is the conjugate semi-axis.

21. Draw the ordinate PN; then, since the hyperbola is rectangular,

$$PN^2 = CN^2 - CA^2 \text{ (\textit{Prop.} X.)}$$
$$\therefore\ PN^2 + AC^2 = CN^2,$$
$$\text{or } AQ^2 = PQ^2.$$
$$\therefore\ AQ = PQ.$$

22. See fig. *Prop.* XXV.

Let CP and CD be a pair of conjugate diameters, which will be equal, since the hyperbola is rectangular.

Let QV, drawn parallel to CD, pass through the focus S, and let it meet the hyberbola again in q.

Through S draw the chord $SQ'V'q'$ parallel to CP, meeting both branches of the hyperbola, and intersecting CD' in V'.

Draw the ordinates PN, $D'M'$ at right angles to the transverse axis, and let the tangents at P and D' meet this axis in the points T and T' respectively; then

$$CV : CP :: CS : CT.$$

But $CS \,.\, CX = CA^2 = CN \,.\, CT$ (*Props.* II. and VIII.)

$$\therefore\ CS : CT :: CN : CX,$$

Hence $CV^2 : CP^2 :: CN^2 : CX^2 \,\ldots\ldots\, (1).$

Again $CV' : CD' :: CS : CT'.$

But $CT' \,.\, CM' = CA^2$ (*Prop.* XV.)

$$= CS \,.\, CX \text{ (\textit{Prop.} II.)}$$
$$\therefore\ CS : CT' :: CM' : CX,$$

Hence $CV'^2 : CD'^2 :: CM'^2 : CX^2 \,\ldots\ldots\, (2).$

$\therefore$ from (1) and (2),

$$CV^2 - CV'^2 : CP^2 :: CN^2 - CM'^2 : CX^2,$$
$$:: CN^2 - PN^2 : CX^2 \text{ (\textit{Prop.} XVI.)}.$$
$$:: AC^2 : CX^2 \text{ (\textit{Prop.} X.)}$$
$$:: CS^2 : CA^2 \text{ (\textit{Prop.} II.)}$$

But $$CS^2 = 2CA^2,$$
$$\therefore CV^2 - CV'^2 = 2CP^2,$$
$$\therefore CV^2 - CP^2 = CP^2 + CV'^2.$$

But $$QV^2 = CV^2 - CP^2 \text{ (\textit{Prop.} XXVI.)}$$
and $$Q'V'^2 = CV'^2 + CP^2 \text{ (\textit{Prop.} XXVI. \textit{Cor.})}$$
$$\therefore QV = Q'V',$$
$$\text{or } Qq = Q'q'.$$

23. Since the hyperbola is rectangular,

$$\therefore PN^2 = CN . NT \text{ (\textit{Prop.} X.)}$$
$$\therefore PN : NT :: CN : PN,$$
$$\therefore \text{the angle } PTN = \text{the angle } CPN,$$
$$\therefore \text{the angle } TCY = \text{the angle } PCN.$$

Hence $$CY : CT :: CN : CP,$$
$$\therefore CY . CP = CN . CT,$$
$$= CA^2 \text{ (\textit{Prop.} VIII.)}$$
$$\therefore CY : CA :: CA : CP,$$
$$\therefore \text{the triangles } CYA,\ CAP \text{ are similar.}$$

24. See fig. *Prop.* XVII.

Let Q be the centre of the circle.

Draw QF at right angles to CO, and let the latus rectum SL be produced to meet the asymptote CR in H; then

$$QF : QC :: CA : CO,$$
$$:: BC : CS.$$

$$\therefore QF : QF + QC :: BC : BC + CS;$$

or $$QF : BC :: AC : B'S_, \ldots . (1);$$

where $S_,$ is the focus of the conjugate hyperbola (Art. 56).

Again, $LS^2 : CS^2 - CA^2 :: BC^2 : AC^2$, (*Prop.* X.)

or $$LS^2 : BC^2 :: BC^2 : AC^2,$$

$$\therefore LS : BC :: BC : AC,$$

and $$HS : BC :: CS : AC,$$

$$\therefore HL : BC :: CS - BC : AC,$$

$$:: BS_, : AC \ldots . (2).$$

Now $$BS_, . B'S_, = CS^2 - CB^2,$$

$$= AC^2,$$

$$\therefore AC : B'S_, :: BS_, : AC.$$

Hence $$QF : BC :: HL : BC,$$

$$\therefore QF = HL.$$

25. See fig. *Prop.* XIX.

Let the chords be drawn parallel to CR; then

$$CH . HR : RQ . QR' :: CR^2 : RR'^2,$$

or $CH . CH' : PL^2 :: CR^2 : 4RV^2$ (*Prop.* XX.)

$$:: CL^2 : 4PL^2,$$

$$\therefore 4CH . CH' = CL^2 = 4CK^2,$$

$$\therefore CH . CH' = CK^2.$$

26. $$RK : RP :: P'K' : P'R',$$

but $PR = P'R'$ (*Prop.* XIX.)

$$\therefore RK = P'K',$$

so also $R'K' = PK.$

27. Draw RM perpendicular to AA' produced; then

$$RM : PN :: AM : AN,$$

and $$RM : QN :: A'M : A'N,$$

$$\therefore RM^2 : PN . QN :: AM . A'M : AN . A'N,$$

$$\text{but } PN \cdot QN = AN \cdot A'N,$$
$$\therefore RM^2 = AM \cdot A'M.$$

$\therefore$ the locus of R is a rectangular hyperbola, of which AA' is the transverse axis.

28. See fig. *Prop.* XXIII.

Let P be one of the points in which the hyperbola in the figure intersect the hyperbola whose axes are in the directions of CL and Cl'; then, since the hyperbolas are rectangular and equal, it is evident that

$$CP \text{ bisects the angle } RCN.$$

$$\begin{aligned} \text{Now the angle } CPT &= \text{the angle } PCD, \\ &= 2 \,.\, \text{the angle } PCH, \\ &= \text{the angle } RCN, \\ &= \text{half a right angle.} \end{aligned}$$

$\therefore$ the angle between the tangents to the hyperbolas at P is a right angle.

29. Join $A'P$, and draw PK at right angles to $A'P$, meeting the transverse axis in K; then

$$\begin{aligned} \text{since } PN^2 : AN \cdot A'N &:: BC^2 : AC^2 \text{ (\textit{Prop.} X.)} \\ \therefore A'N \cdot NK : AN \cdot A'N &:: BC^2 : AC^2, \\ \text{or } NK : AN &:: BC^2 : AC^2, \\ &:: QP : AQ, \end{aligned}$$
$$\therefore NQ \text{ is parallel to } PK.$$

$\therefore$ also QH drawn at right angles to QN is parallel to $A'P$.

Hence $\qquad AH : HA' :: AQ : QP.$

30. Let SH be the given base; and P, Q the points of trisection of a segment described upon it.

It is evident that PQ is parallel to SH.

Bisect SH in X, and draw XM perpendicular to PQ, bisecting it in M; then

$$SP = PQ = 2PM.$$

$\therefore$ the locus of P is a hyperbola whose focus is S, and directrix XM.

31. Since the triangles SCs, TCt are equal (*Prop.* XXII.)

$\therefore$ also the triangles TVS, tVs are equal,

and the $\angle SVT$ is equal to the $\angle sVt$,

$\therefore VS : Vs :: Vt : VT$ (*Euclid*, VI. 14.)

32. See fig. *Prop.* XII.

Let $S'Y'$ meet the auxiliary circle in Z'; and draw $Z'M'$ perpendicular to $S'P$, then

$$S'M' : S'Z' :: S'Y' : S'P,$$

$$\therefore S'M' \cdot S'P = S'Z' \cdot S'Y',$$

$$= BC^2 \text{ (\textit{Prop.} XII.)}$$

$\therefore Z'M'$ is the chord of contact of the pair of tangents drawn from P to the circle whose centre is S', and radius BC.

Again, since CY produced passes through Z', and CY is parallel to $S'P$,

$\therefore$ the $\angle CZ'M'$ is a right angle,

$\therefore Z'M'$ touches the auxiliary circle.

33. (1) Let C be between A and B; and let the tangents AQ, BR meet in P; then

$$AP \backsim PB = AQ \backsim BR,$$

$$= AC \backsim BC.$$

Therefore the locus of P is a hyperbola, of which A and B are the foci; unless AC is equal to BC, when the locus is a straight line bisecting AB at right angles.

(2) Let C be on AB produced, and let the tangents AQ, BR as before meet in P; then

$$AP + BP = AQ + BR,$$

$$= AC + BC,$$

$\therefore$ the locus of P is an ellipse.

34. Draw QM perpendicular to AA'; then

$$QM : AM :: PN : AN,$$
$$\text{and } QM : A'M :: P'N : A'N,$$
$$\therefore QM^2 : AM . A'M :: PN^2 : AN . A'N,$$
$$:: BC^2 : AC^2.$$

$\therefore$ the locus of Q is a hyperbola, of which AC and BC are the semi-axes.

35. Since $\left.\begin{array}{l} PF . CD = AC . BC, \\ \text{and } PF . PG = BC^2. \end{array}\right\}$ (*Prop.* XXV. *Cor.*)

$$\therefore CD : PG :: AC : BC;$$

hence, when the hyperbola is rectangular,

$PG = CD = PL$ (*Prop.* XXIII.)

and $PL = Pl$ (*Prop.* XIX. *Cor.*)

$\therefore$ the angle LGl is a right angle.

36. Let A be the common vertex, and NPQ a common ordinate to the ellipse and parabola; then

$$PN^2 : AN . A'N :: BC^2 : AC^2,$$

or denoting the diameter of the circle of curvature at A by $2R$.

$PN^2 : AN . A'N :: R . AC : AC^2$ (*Prop.* XXVI. *Ellipse*)

$$:: R : AC,$$
$$:: 2R . AN : 2AC . AN,$$
$$\therefore PN^2 : 2R . AN :: AN . A'N : 2AC . AN.$$
$$:: A'N : 2AC.$$

But $QN^2 = 4AS . AN,$

$= 2R . AN$ (*Prop.* XIX. *Parabola*)

$$\therefore PN^2 : QN^2 :: A'N : 2AC.$$

Now in the ellipse $A'N$ is less than $2AC$,

$$\therefore PN \text{ is } < QN.$$

Again, if PN is an ordinate of the hyperbola, the same demonstration applies, except that

$$A'N \text{ is } > 2AC,$$
$$\text{and } \therefore PN > QN.$$

37. Produce RQ to meet the asymptote in r, and draw PL parallel to CR', meeting CR in L; then

since Rr is bisected in Q (*Prop.* XIX. *Cor.* 1)

$$\begin{aligned}
\therefore\ 2QK &= CR, \\
&= CL + RL. \\
\text{But } RL &= P'H', \\
\text{since } RP &= R'P' \quad (\textit{Prop.}\ \text{XIX.}) \\
\therefore\ 2QK &= CL + P'H', \\
&= PH + P'H'.
\end{aligned}$$

38. Draw PH, $P'H'$ parallel to one asymptote, and PK $P'K'$ parallel to the other.

Let PH, $P'K'$ meet in Q', and $P'H'$, PK in Q; then

$$PH \,.\, PK = P'H' \,.\, P'K' \quad (\textit{Prop.}\ \text{XXI.})$$
$$\therefore\ PH : P'K' :: P'H' : PK,$$
$$\text{or } CK : KQ :: CK' : K'Q',$$
$$\therefore\ QQ' \text{ passes through } C.$$

39. Draw PH, PK parallel to the sides CF, CE of the rectangle; then

$$4PH \,.\, PK = CE \,.\, CF,$$

and is therefore constant.

Hence the locus of P is a rectangular hyperbola, of which the sides of the rectangle are asymptotes.

40. Let the tangent at P meet the asymptotes CM, CN in L and l.

Join MN; then

PM and PN are tangents to the ellipse (*Art.* 36, *Def.*)

$\therefore$ the tangent at Q is parallel to MN (*Prop.* XVIII. *Ellipse.*)

Also, since Ll is bisected in P (*Prop.* XIX. *Cor.* 1.)

$\therefore$ CL and Cl are bisected in M and N,

$\therefore$ Ll is parallel to MN,

$\therefore$ the tangents at P and Q are parallel.

41.
$$NP \,.\, NQ = AN \,.\, A'N,$$
$$\therefore PN^2 : NP \,.\, NQ :: BC^2 : AC^2,$$
$$\text{or } PN : QN :: BC^2 : AC^2,$$
$$:: BC : B'C,$$

if $B'C$ be a third proportional to BC and AC,
$$\therefore QN^2 : PN^2 :: B'C^2 : BC^2,$$
$$\text{and } PN^2 : AN \,.\, A'N :: BC^2 : AC^2,$$
$$\therefore QN^2 : AN \,.\, A'N :: B'C^2 : AC^2,$$

$\therefore$ the locus of Q is a hyperbola, of which AC and $B'C$ are the transverse and conjugate semi-axes.

42. See fig. *Prop.* XXV.

Let PP' be the diameter, and Q any point on the curve.

Join PQ, $P'Q$, and draw CU parallel to $P'Q$, bisecting PQ in the point U.

Then if CU' be drawn parallel to PQ, CU and CU' will be conjugate diameters, and therefore (see *Prob.* 15) will make equal angles with CR, as also with CP and CD.

$$\text{Hence } \angle QPP' - \angle QP'P = \angle U'CP' - \angle UCP,$$
$$= \angle U'CP' - \angle U'CD,$$
$$= \angle P'CD.$$

43. Let TPt be a common tangent to the hyperbola and any one of the elliptic quadrants APB, and let it meet the asymptotes CA, CB in T and t.

Draw the ordinates PN, Pn parallel respectively to Ct and CT; then

since TPt is bisected in P (*Prop.* XIX. *Cor.* 1.)

$$\therefore CT \text{ is bisected in } N.$$

But from the ellipse $AC^2 = CN \,.\, CT$,

$$\therefore AC^2 = 2CN^2.$$

So also $$BC^2 = 2Cn^2,$$

$$\therefore AC^2 : BC^2 :: CN^2 : Cn^2,$$

$$\therefore AC : BC :: CN : Cn,$$

$$\therefore AC \,.\, BC : BC^2 :: CN \,.\, Cn : Cn^2,$$

$$\therefore AC \,.\, BC = 2CN \,.\, Cn,$$

$$= \tfrac{1}{2}CS^2. \quad (\textit{Prop.}\ \text{XXI.})$$

44. Produce OP to meet CD, the diameter conjugate to CP, in F; then

$$OP : PQ :: CP : PF,$$

$$\therefore OP \,.\, PF = PQ \,.\, CP,$$

$$= CP^2,$$

$\therefore$ O is the centre of the circle of curvature at P. (*Prop.* XXIX.)

45. See fig. *Prop.* XXV.

Let Cl and CL be conjugate diameters of an ellipse, with C as centre, described touching the hyperbola, of which Cl and CL are asymptotes, in P.

Draw the common tangent LPl, and from the point L draw LDl', touching the conjugate hyperbola in D; then

CP and CD are conjugate diameters of the hyperbola, and PD is parallel to ll'. (*Prop.* XXIII.)

Join PD, meeting CL in M; then

since PD is a parallelogram,

$\therefore MD = PM.$

Hence, since CL and Cl' are conjugate diameters of the ellipse,

$\therefore D$ is a point on the ellipse;

and, since the tangent to the ellipse at P is parallel to CD,

$\therefore$ the tangent to the ellipse at D is parallel to CP,

$\therefore LD$ touches the ellipse at D,

or the ellipse and hyperbola have a common tangent at D, and therefore touch one another.

Also CP and CD are conjugate diameters of the ellipse, as well as of the hyperbola.

46. Let Q, Q' be the points where the common tangents QP, $Q'P'$ touch the hyperbola, and P, P' the points where they touch the ellipse.

Through O, the point of intersection of QP and $Q'P'$, draw CO, meeting PP' and QQ' in U and V respectively.

Join CQ, CQ' meeting the ellipse in q, q'; then

since QQ', PP' are bisected in V and U, they are evidently parallel.

Now, since a line drawn through q, parallel to PP' and QQ', will be bisected by CUV, whether it be terminated by the ellipse or CQ', it is evident that

qq' is parallel to QQ'.

Hence $CQ : CQ' :: Cq : Cq'$.

$\therefore$ an ellipse similar to the given ellipse, and having also C' for its centre, can be described through the points Q and Q'.

47. Let PO, $P'O'$ be the radii of curvature at P and P' respectively; and let PF, $P'F'$ be the perpendiculars from P and P', upon the diameters CD and CD', conjugate respectively to CP and CP'; then

$$PO \,.\, PF = CD^2 \text{ (\textit{Prop.} XXIX.)}$$

$$\therefore\ PO \,.\, PF \,.\, CD = CD^3,$$

$$\text{or } PO \,.\, AC \,.\, BC = CD^3 \text{ (\textit{Prop.} XXV. \textit{Cor.})}$$

So also $P'O' \,.\, AC \,.\, BC = CD'^3$,

$$\therefore\ PO : P'O' :: CD^3 : CD'^3,$$

$$:: CP^3 : CP'^3 \text{ (\textit{Prob.} 13)},$$

since the hyperbola is rectangular.

48. Let R and S be two of the points of intersection on the same side of PQ; and let PQ intersect the hyperbola in p and q.

Draw OV bisecting PQ in V; then

since $Pp = Qq$ (*Prop.* XIX.)

$\therefore$ V is also the middle point of pq.

Now, since the chords which are drawn parallel to PQ are bisected both in the ellipse and hyperbola by OV, it is evident that the chord drawn from R, parallel to PQ, must meet the ellipse and hyperbola in the same point.

Hence RS is parallel to PQ.

So also, if R', S' be the points of intersection of the hyperbola and ellipse on the other side of PQ,

$R'S'$ is parallel to PQ.

Produce PR to meet OV in T, and produce TS to meet PQ in Q'; then if RS meet OV in M,

$$Q'V : TV :: SM : TM,$$

$$:: RM : TM,$$

$$:: PV : TV,$$

$$:: QV : TV,$$

$\therefore$ Q and Q' coincide,

and TS passes through the point Q.

Hence, since the tangents to the hyperbola at R and S meet OV in the same point (*Prop.* XXVII. *Cor.*), and PR

and QS have also been proved to intersect on OV, it is evident that,

if PR is the tangent at R,

QS must also be the tangent at S.

Lastly, whether U be the point where RQ or PS' intersects OV,

$$MU : UV :: RM : PV,$$

$\therefore$ RQ and PS meet on the line OV, which bisects PQ.

49. Let the points A and B be on the same branch of the hyperbola.

Let AQ and BP intersect in R, join CR, and produce CR to meet AB in O; then, since the angles at P and Q are right angles, and the three perpendiculars from the angles of a triangle on the opposite sides meet in a point,

$\therefore$ CR is at right angles to AB.

Now by similar triangles

$$CO : OA :: BO : OR,$$

$$\therefore CO \, . \, OR = OA \, . \, BO.$$

Hence, by the converse of the *Cor.* of *Prop.* XXVIII., since the hyperbola is rectangular, C is also a point on the hyperbola, but situated on the opposite branch to that on which AB is drawn, and on which C moves.

PROBLEMS ON THE SECTIONS OF THE CONE.

1. See fig. *Prop.* I.

Let C be the centre of the sphere inscribed in the cone, and touching the plane of the parabolic section in the focus S.

Join CA, CE; then

since AS is parallel to Oe,

it is evident that

the $\angle ACO$ is a right angle.

Now $\quad SA : AO :: AE : AO.$

But since $\quad AE : AC :: AC : AO,$

$\therefore AE : AO :: AC^2 : AO^2.$

Hence $\quad SA : AO :: AC^2 : AO^2,$

$\therefore$ the ratio of SA to AO is independent of the position of the point A, and depends only upon the vertical angle of the cone.

The foci therefore of all parabolic sections made by planes perpendicular to the plane of the paper, will be upon a straight line drawn through O; and the foci of all the parabolic sections that can be made by any planes, upon the surface of the cone formed by the revolution of this line round the axis of the given cone.

2. See fig. 2, *Prop.* I.

$$CS : CA :: SA : AX,$$
$$:: AE : AX.$$

Now if the ratio of AE to AX is fixed,

since AEX is a fixed angle,

and AE is less than AX,

and AXE is therefore less than a right angle;

the $\angle AXE$ is fixed, and can have only one value (*Euclid*, VI. 7).

$$\therefore \text{ also the } \angle SAO \text{ is fixed.}$$

Draw AC' to the centre of the inscribed sphere bisecting the angle SAO; then the angle $C'AE$ is a fixed angle.

Now $\qquad SA : AO :: AE : AO,$

and the ratio of AE to AO is compounded of the ratios of $AE : AC'$ and $AC' : AO$; but since the $\angle C'AE$ is fixed, the ratios of $AE : AC'$, and $AC' : AO$ are both fixed ratios.

Hence $SA : AO$ is a fixed ratio.

Therefore, as in the preceding problem, the locus of S is a cone, having O for its vertex.

In the same manner it may be shown that the other focus S' will always lie on another cone which has also O for its vertex.

3. See fig. 2, *Prop.* I.

Let AA' be the major axis of one of the ellipses.

Since the cutting planes are parallel, the major axes of the elliptic sections will be parallel also.

Hence, if O be joined with the middle point of AA', the line so drawn will also pass through the centres of all the other elliptic sections.

If through this line a plane be drawn perpendicular to the plane of the paper, the extremities of the minor axes will all lie on the two lines in which this plane intersects the surface of the cone.

4. See fig. 1, *Prop.* I.

$$\text{The latus rectum} = 4AS,$$

and in Prob. 1 it has been shown that AS bears a ratio to AO, depending only upon the vertical angle of the cone.

Hence the latus rectum varies as AO.

5. See fig. 3, *Prop.* I.

The angle KOL represents half the angle between the asymptotes of the section made by a plane parallel to KOL.

Now, in the rectangular hyperbola, the angle between the asymptotes is a right angle, and the greatest value which KOL admits of is half the vertical angle of the cone.

Hence, if the vertical angle of the cone is less than a right angle, it will be impossible to cut it in such a manner that the section may be an equilateral hyperbola.

When the vertical angle is a right angle, any section made by a plane parallel to a plane through the axis will intersect the curve in a rectangular hyperbola.

When the angle ROr is greater than a right angle, bisect OR in V, and draw VZ at right angles to OR, making VZ equal to OV, and join OZ; then

$$OR^2 = 2\,OZ^2.$$

With centre O, and radius OZ, describe a circle, cutting Rr in L; then

$$\begin{aligned} KL^2 &= OK^2 - OL^2, \\ &= OR^2 - OL^2, \\ &= 2\,OL^2 - OL^2, \\ &= OL^2. \end{aligned}$$

Hence the angle KOL is half a right angle, and therefore any plane parallel to KOL will intersect the cone in an equilateral hyperbola.

6. When the cutting plane is parallel to OI and Oi, the perpendicular from O on the cutting plane is the conjugate semi-axis of both hyperbolas.

The latera recta are therefore inversely proportional to the transverse axes. (Art. 48.)

Also OD, Od, and OD, Od' are parallel to the asymptotes of the hyperbolas in the cones whose axes are respectively OI and Oi.

Let Oa, ab, measured along and perpendicular to OI, represent the semi-axes of the one hyperbola, and Oa', $a'b'$,

measured along and perpendicular to Oi, the semi-axes of the other hyperbola.

Then $$Oa : ab :: OI : Oi,$$
and $$a'b' : Oa' :: OI : Oi;$$
$\therefore$ since $$ab = a'b',$$
$$Oa : Oa' :: OI^2 : Oi^2.$$
$\therefore$ the latera recta are as $Oi^2 : OI^2$.

Next, when the cutting plane is perpendicular to OI and Oi, the transverse axis of the hyperbola is equal to the major axis of the ellipse.

Let AA' be the common axis (see fig. 3, *Prop.* I.), C the common centre; then, if Cuu' be drawn parallel to OI, meeting OD, Od' in u and u'; then

the latera recta are as $CU \,.\, CU' : Cu \,.\, Cu'$ (*Prop.* I.)

as $Oi^2 : OI^2$.

APPENDIX.

1. The following is an independent solution of *Prob.* 26 of the Parabola.

Join AP, AQ, and let them meet in p and q, the line drawn through O at right angles to the axis; then, since

$$Op : PM :: AO : AM,$$
$$\text{and } PM : 4AS :: AM : PM,$$
$$\therefore Op : 4AS :: AO : PM.$$

So also
$$Oq : 4AS :: AO : QN,$$
$$\therefore Op : Oq :: QN : PM,$$
$$:: QO : PO,$$
$$\therefore pQ \text{ is parallel to } Pq,$$
$$\therefore Ap : AP :: AQ : Aq,$$
$$\therefore AO : AM :: AN : AO,$$
$$\therefore AO^2 = AM \,.\, AN.$$

2. The following proof of the more general property, of which *Prob.* 44 of the ellise isa particular case, is deserving of notice.

Let O be any point. It is required to find the locus of the middle points of chords drawn through O.

Join OC, and produce it to meet the ellipse in K and K'.

Draw *any* chord POp, and let Q be its middle point, and through K draw the chord KR parallel to POp. Produce CQ to meet KR in r; then KR is bisected in r.

Hence
$$OQ : Kr :: OC : CK,$$
$$\therefore OQ : KR :: OC : KK',$$

i.e. OQ bears a constant ratio to KR,

$\therefore$ the locus of Q is an ellipse similar to the given ellipse.

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